

HOMOTOPICAL INTERSECTION THEORY, III: MULTI-RELATIVE INTERSECTION PROBLEMS

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ABSTRACT. This paper extends some results of Hatcher and Quinn [HQ] beyond the metastable range. We give a bordism theoretic obstruction to deforming a map $f: P \rightarrow N$ between manifolds simultaneously off of a collection of pairwise disjoint submanifolds $Q_1, \dots, Q_j \subset N$. In a certain range of dimensions, our obstruction is the entire story.

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1. INTRODUCTION

In [KW1] we considered the problem of deforming a map $f: P \rightarrow N$ between compact smooth manifolds off a compact smooth submanifold $Q \subset N$. This was called an *intersection problem* in [KW1]. We obtained there an obstruction $\chi(f)$ residing in a normal bordism group $\Omega_0(X; \xi)$ whose vanishing is necessary for finding such a deformation. In a certain metastable range of dimensions, $\chi(f)$ was shown to be the total obstruction. The goal of this paper is to extend these ideas to the multi-relative setting.

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For a positive integer j , let $Q_1, \dots, Q_j \subset N$ be a collection of pairwise disjoint compact smooth submanifolds of a compact connected smooth manifold N . Here N is allowed to have a boundary, but each Q_i is assumed to have empty boundary. Given a map $P \rightarrow N$, where P is a compact manifold, the problem we consider is to find a deformation of f off of the Q_i *simultaneously*. We approach this inductively, by assuming that P can be deformed off of any proper union of the Q_i in such a way that the deformations line up in a certain fashion.

To make this precise, recall that a $(k+1)$ -ad of spaces consists of a space X together with k subspaces $X_1, \dots, X_k \subset X$. The notation for such data is $(X; X_1, \dots, X_k)$, but we will sometimes simply write X when the subspaces are understood.

Example 1.1. (1). A space Z can be considered as a constant $(k+1)$ -ad, i.e., $(Z; Z, \dots, Z)$.

(2). The standard $(k-1)$ -simplex Δ^{k-1} together with its codimension one faces is a $(k+1)$ -ad, i.e., $(\Delta^{k-1}, d_0\Delta^{k-1}, \dots, d_{k-1}\Delta^{k-1})$.

(3). If Z is a space and X is a $(k+1)$ -ad, then the cartesian product $Z \times X$ is a $(k+1)$ -ad in the evident way.

A map of $(k+1)$ -ads $X \rightarrow Y$ is a continuous map of underlying spaces which restricts to maps $X_i \rightarrow Y_i$ for all i . We can topologize these as the subspace of all maps from X to Y in the compact-open topology.

Let us now consider N together with the subspaces $N - Q_1, \dots, N - Q_j$ as a $(j+1)$ -ad: $(N; N - Q_1, \dots, N - Q_j)$. Then a *multi-relative intersection problem* is defined to be a map of $(j+1)$ -ads

$$f: P \times \Delta^{j-1} \rightarrow N.$$

Let us consider the space $N - Q_{\mathcal{J}}$ as constant $(j+1)$ -ad; it is then a sub-ad of N . We define a *solution* to our problem to be a homotopy of maps of $(j+1)$ -ads from f to $g: P \times \Delta^{j-1} \rightarrow N$ such that g has image in $N - Q_{\mathcal{J}}$.

Example 1.2. Consider the case $j = 2$. Then a multi-relative intersection problem in this instance amounts to a homotopy $f_t: \rightarrow N$ such that f_0 has image in $N - Q_1$ and f_1 has image in $N - Q_2$.

One can think of f_t for $0 \leq t \leq 1/2$ as a deformation of $f_{1/2}$ off of Q_1 and f_t for $1/2 \leq t \leq 1$ as a deformation of $f_{1/2}$ off of Q_2 . A solution to the intersection problem gives a 1-parameter family of such deformations $f_{s,t}$ for $s, t \in [0, 1]$ with the property that $f_{0,t} = f_t$ and $f_{1,t}: P \rightarrow N$ has image disjoint from Q_{12} . In particular, we can think

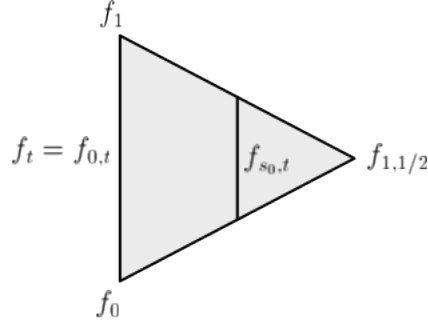


FIG. 1. Schematic of a solution to a multi-relative intersection problem in the $j = 2$ case. The map $f_t = f_{0,t}$ encodes the original problem, which is a deformation from a map missing Q_1 to a map missing Q_2 . For each fixed $s_0 \in [0, 1]$, the family $f_{s_0,t}$ is also a deformation from a map missing Q_1 to a map missing Q_2 . The map $f_{1,1/2}$ misses both Q_1 and Q_2 .

of $f_{s,1/2}$ as a deformation of $f_{1/2}$ to a map $f_{1,1/2}$ whose image is disjoint from Q_{12} (cf. Fig. 1).

In modern terminology we can reformulate the problem as follows: let $\mathcal{J} = \{1, \dots, j\}$. If $S \subset \mathcal{J}$, let

$$Q_S = \coprod_{i \in S} Q_i.$$

Then a multi-relative intersection problem is equivalent to specifying a map

$$f: P \rightarrow \operatorname{holim}_{S \subsetneq \mathcal{J}} (N - Q_S),$$

where the target is the homotopy inverse limit of the spaces $N - Q_S$ as S ranges through the proper subsets of \mathcal{J} . Then the goal is to find a lift

$$\begin{array}{ccc} & N - Q_{\mathcal{J}} & \\ & \downarrow & \\ P & \xrightarrow{\quad} & \operatorname{holim}_{S \subsetneq \mathcal{J}} (N - Q_S) \end{array}$$

which makes the diagram homotopy commute.

Given the above data, we write

$$E(P, Q_{\bullet})$$

for the *iterated homotopy fiber product* of $P \times \Delta^{j-1}$ and each of the Q_i over N . It is defined to be the homotopy pullback of the diagram

$$P \times \Delta^{j-1} \times \prod_{i=1}^j Q_i \longrightarrow \prod_{i=0}^j N \xleftarrow{\Delta} N$$

where Δ is the diagonal map, and the left vertical map is the product of f with the inclusions of the Q_i .

Define a virtual bundle ξ over $E(P, Q_\bullet)$ as follows: Let τ_P be the tangent bundle of P , τ_N the tangent bundle of N and τ_{Q_i} the tangent bundle of Q_i ; each one of these gives a bundle over $E(P, Q_\bullet)$ using the evident (projection) maps. To avoid notational clutter, we use the same notation for these pullbacks. Then

$$\xi := -\tau_P + \sum_{i=1}^j (\tau_n - \tau_{Q_i}).$$

Suppose $p = \dim P$, $q_i = \dim Q_i$ and $n = \dim N$. It will convenient to write

$$\mu = \min_i (n - q_i - 2) \quad \text{and} \quad \Sigma = \sum_i (n - q_i - 2).$$

In particular, the virtual rank of ξ is $2j - p + \Sigma$.

Theorem A. *Assume $j \geq 1$. Then there is an obstruction*

$$\chi(f) \in \bigoplus_{(j-1)!} \Omega_{2j-2}(E(P, Q_\bullet); \xi)$$

which vanishes if f can be homotopy factorized through $N - Q_{\mathcal{J}}$.

Conversely, if

$$p \leq 1 + \mu + \Sigma$$

then the vanishing of $\chi(f)$ is also sufficient for finding such a factorization of f .

Remark 1.3. The obstruction $\chi(f)$ is defined in a homotopy theoretic manner, and its interpretation as an element in a direct sum of bordism groups relies on the Pontryagin-Thom isomorphism between bordism and the homotopy groups of the associated Thom spectrum. It is therefore reasonable to ask what $\chi(f)$ means geometrically. A class in the obstruction group appearing in Theorem A is represented by a $(j-1)!$ -tuple $(D_1, \dots, D_{(j-1)!})$ in which each D_i is a certain closed manifold of dimension $p - 2 - \Sigma$, equipped with a map $D_i \rightarrow E(P, Q_\bullet)$ such that the pullback of ξ is identified with the stable normal bundle. When $j = 1$, this manifold can be taken as the transverse intersection of $f: P \rightarrow N$ with the submanifold $Q \subset N$ (see [KW1, th. 12.1]).

We do not have as yet a bordism description of $\chi(f)$ for $j > 1$, but here is a conjectural one when $j = 2$. Let $A_i \subset Q_i$ be the transverse intersection of $f: P \rightarrow N$ with Q_i . By assumption, $f: A_i \rightarrow E(P, Q_i)$ is null bordant. Let $g_i: W_i \rightarrow E(P, Q_i)$ be a null-bordism. compose this with the projection $E(P, Q_i) \rightarrow P$ to get maps $h_i: W_i \rightarrow P$. Now

take the transverse intersection of $h_1 \times h_2: W_1 \times W_2 \rightarrow P \times P$ with the diagonal of P . This produces a closed manifold D of dimension $p - 2 - \Sigma$, which comes equipped with a map $D \rightarrow E(P, Q_\bullet)$ which is covered by the requisite bundle data. We conjecture that the associated bordism class coincides with the obstruction $\chi(f)$.

Remark 1.4. An alternative way to write the obstruction group is to set $\xi' = \xi - (2j - p - 2)\epsilon$, where ϵ is the trivial bundle of rank one. Then ξ' has virtual rank 0. One then has a preferred identification of bordism groups $\Omega_{p-2-\Sigma}(E(P, Q_\bullet); \xi') \cong \Omega_{2j-2}(E(P, Q_\bullet); \xi)$.

Remark 1.5. The integer $(j-1)!$ appearing in Theorem A is reminiscent of the number of μ -invariants of classical links with j components (cf. Milnor [Mi]). In [KW1, §9] we described linking invariants of link maps $f: P \amalg Q \rightarrow N$, where P and Q are closed manifolds and N is an arbitrary compact manifold, possibly with boundary (where a link map f is a map which satisfies the condition $f(P) \cap f(Q) = \emptyset$).

It turns out that one can use the ideas of this paper to construct linking invariants of link maps $f: Q_1 \amalg \cdots \amalg Q_j \rightarrow N$ (i.e., a map such that $f(Q_i) \cap f(Q_{i'}) = \emptyset$ for $i \neq i'$). These invariants lie in a direct sum of $(j-1)!$ -copies of a suitable bordism group, giving a generalization of Milnor's invariants to arbitrary manifolds.

We originally had planned to pursue these ideas in separate paper, but while the current paper was being written, Brian Munson, using an altogether different approach, constructed a family of such generalized linking invariants in the case when N is a closed disk [Mu, th. 1.1]. It seems to us that his approach should also handle the case of a general N .

For dimensional reasons, the bordism group appearing in Theorem A vanishes when $p \leq 1 + \Sigma$, hence

Corollary B. *Assume $p \leq 1 + \Sigma$. Then f can be homotopy factorized through $N - Q_{\mathcal{J}}$.*

Remark 1.6. When $j = 1$, the corollary says that a map $f: P \rightarrow N$ can be deformed off of a submanifold $Q \subset N$ provided that $p + q + 1 \leq n$. This follows from transversality.

When $j > 1$, transversality does not imply the corollary, at least not directly. In this situation, however, the corollary follows more directly from the generalized Blakers-Massey theorem applied to the j -cubical diagram $\{N - Q_S\}_{S \subset \mathcal{J}}$.

Highly connected manifolds. When the manifolds P and Q_i are sufficiently highly connected, the obstruction group of Theorem A has a

simpler description up to isomorphism. Suppose that P is a -connected and Q_i is b_i -connected. Choose basepoints in $x \in P$ and $y_i \in Q_i$. Then x gives rise to a point $x' \in N$ using f . The the homotopy fiber product of $E(x, y_\bullet)$ is defined and comes equipped with a map $E(x, y_\bullet) \rightarrow E(P, Q_\bullet)$. Moreover, the pullback of ξ to $E(x, y_\bullet)$ is a trivial virtual bundle of rank $2j - p + \Sigma$. Hence the bordism groups associated with this pullback are framed bordism groups of $E(x, y_\bullet)$ shifted in degree by $2j - p + \Sigma$.

It is also straightforward to check that the map $E(x, y_\bullet) \rightarrow E(P, Q_\bullet)$ is $\min(a, b_1, \dots, b_j)$ -connected. It follows that the associated map of Thom spectra is k -connected, where $k = \min(a, b_1, \dots, b_j) + 2j - p + \Sigma$. In particular the induced homomorphism of bordism groups is an isomorphism in degrees strictly less than k .

The space $E(x, y_\bullet)$ consists of j -tuples $(\lambda_1, \dots, \lambda_j)$ such that $\lambda_i: [0, 1] \rightarrow N$ is a path from x' to y_i for $1 \leq i \leq j$. The j -fold cartesian product of loop spaces $\prod_j \Omega N$ based at x' acts on $E(x, y_\bullet)$ by path composition. After a basepoint $E(x, y_\bullet)$ is chosen, we obtain a homotopy equivalence $E(x, y_\bullet) \simeq \prod_j \Omega N$. Consequently, we have shown

Corollary C. *Assume $p \leq 1 + \Sigma + \min(a, b_1, \dots, b_j)$. Then the obstruction group appearing in Theorem A is isomorphic to the direct sum of framed bordism groups*

$$\bigoplus_{(j-1)!} \Omega_{p-2-\Sigma}^{\text{fr}} \left(\prod_j \Omega N \right).$$

Example 1.7. Suppose $P = S^p$ and $Q_i = S^{q_i}$ are spheres. Then $a = p-1$ and $b_i = q_i - 1$. Consequently, the inequality appearing in Corollary C becomes $p \leq \Sigma + \mu - j$.

Example 1.8. Suppose $p = 2 + \Sigma$ and $a, b_i \geq 1$. Then the obstruction group of Corollary C is isomorphic to $\oplus_{(j-1)!} \mathbb{Z}[\pi]^{\otimes j}$, with $\pi = \pi_1(N)$. Presumably, $\chi(f)$ in this instance is a kind of higher order intersection number.

Embeddings. There is a version of the multi-relative intersection problem that applies to embeddings. In this instance one is given a map of $(j+1)$ -ads $f: P \times \Delta^{j-1} \rightarrow N$ which is also a $(j-1)$ -parameter family of smooth embeddings from P to N . The solution of the problem in this case is to find a deformation of ad-maps, this time through an isotopy, to a $(j-1)$ -parameter family of embeddings having image disjoint from $Q_{\mathcal{J}}$.

Theorem D. *Assume $p, q_i \leq n - 3$ and $p \leq 1 + \min(n - p - 2, \mu) + \Sigma$. Then $\chi(f) = 0$ if and only if the multi-relative intersection problem of embeddings has a solution.*

Theorem A is proved using a version of Poincaré duality together with some general results about strongly cocartesian j -cubes (Corollary 3.14). The results about cubes are too technical to state in the introduction, but is worth mentioning, as it may be of independent interest (see §3 below).

Remark 1.9. The $j = 1$ case (“the metastable range”) of Theorem A was already considered in [KW1]. That work gave a homotopy theoretic approach to the main results of the paper of Hatcher and Quinn [HQ] (when $j = 1$, Theorem D amounts to the vanishing obstruction case of [HQ, th. 2.2]). In addition to Hatcher and Quinn, the papers of Dax [Da] and Salomonsen [Sa] also considered embedding problems in the metastable from the point of view of bordism. A certain thread of these ideas can be traced back to the work of Haefliger [Ha].

A guiding motivation of this work was to find an approach to solving embedding problems. In §8 we will formulate a conjecture for all j which, if valid, gives information about embedding theory outside the metastable range.

2. TERMINOLOGY

Let \mathcal{T} be the category of compactly generated spaces. For $X \in \mathcal{T}$, we let $\mathcal{T}(X)$ denote the category of spaces over X . This is the category whose objects are pairs (Y, r) such that $r: Y \rightarrow X$ is a map. A morphism $(Y, r) \rightarrow (Y', r')$ is a map $f: Y \rightarrow Y'$ such that $r' \circ f = r$. We more often as not suppress the structure map $r: Y \rightarrow X$ when specifying an object and write Y in place of (Y, r) .

Similarly let $\mathcal{R}(X)$ denote the category of retractive spaces over X . This has objects (Y, r, s) where $r: Y \rightarrow X$ and $s: X \rightarrow Y$ are maps such that $r \circ s$ is the identity map. A morphism $(Y, r, s) \rightarrow (Y', r', s')$ is a map $f: Y \rightarrow Y'$ such that $r' \circ f = r$ and $f \circ s = s'$. Again, the structure maps are usually suppressed. Note that the case $\mathcal{R}(*)$ gives the category of based spaces. We sometimes regard objects of $\mathcal{R}(X)$ as objects of $\mathcal{T}(X)$ by means of the forgetful functor.

Both $\mathcal{T}(X)$ and $\mathcal{R}(X)$ have simplicial model category structures where a weak equivalence (cofibration, fibration) in each case is a morphism whose underlying map is a weak homotopy equivalence (cofibration, fibration) of spaces [Q, 2.8, prop. 6] (here we are using the model structure on topological spaces in which a weak equivalence is a weak

homotopy equivalence, a fibration is a Serre fibration, and a cofibration is a map which is a retract of a relative cell complex). In particular, one may define the set of (fiberwise) homotopy classes $[Y, Z]_{\mathcal{T}(X)}$ is defined for objects Y, Z of $\mathcal{T}(X)$. Similarly one may define homotopy classes in $\mathcal{R}(X)$. If $Y \in \mathcal{T}(X)$ is an object, let $Y^+ \in \mathcal{R}(X)$ be the object given by $Y \amalg X$ with evident structure maps. If $Z \in \mathcal{R}(X)$ is an object, then we have $[Y^+, Z]_{\mathcal{R}(X)} = [Y, Z]_{\mathcal{T}(X)}$.

Warning. As usual, when defining homotopy classes $[Y, Z]_{\mathcal{T}(X)}$ it is implicit that Y be replaced by a cofibrant approximation and that Z be replaced by a fibrant approximation.

A morphism $Y \rightarrow Z$ in either $\mathcal{T}(X)$ or $\mathcal{R}(X)$ is said to be *j-connected* if and only if its underlying map in \mathcal{T} of spaces is *j-connected* (i.e., it induces a surjection on homotopy in degrees $\leq j$ and an injection in degrees $< j$, for all choices of basepoint). An object Y is *j-connected* if and only iff the structure map $Y \rightarrow X$ is $(j+1)$ -connected. An object is *weakly contractible* if and only if it is *j-connected* for all j .

A commutative square of $\mathcal{T}(X)$ or $\mathcal{R}(X)$ is *j-cocartesian* (*j-cartesian*) if it is so when considered in \mathcal{T} (here j can be an integer or ∞). An ∞ -cocartesian (∞ -cartesian) square of $\mathcal{T}(X)$

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & C \end{array}$$

in which X' is weakly contractible is said to be a *homotopy cofiber sequence*. By slight abuse of notation, sometimes suppress X' and write it as $A \rightarrow Y \rightarrow C$. More generally, one has the notion of homotopy cofiber sequence in dimensions $\leq s$, which in this instance means that the map $\mathrm{hocolim}(X' \leftarrow A \rightarrow Y) \rightarrow C$ is s -connected. These notions dualize in the obvious way to homotopy fiber sequences.

We say an object Y of $\mathcal{T}(X)$ has dimension $\leq s$ if the underlying space of Y is a cell complex of dimension $\leq s$. In this case we write $\dim Y \leq s$. If Y is an object of $\mathcal{R}(X)$ we define the notion of dimension in a slightly different way: we write $\dim Y \leq s$ in this case if Y is obtained from X by attaching cells of dimension at most s .

Lemma 2.1. *Suppose that $A \rightarrow Y \rightarrow C$ is a homotopy cofiber sequence of $\mathcal{T}(X)$. Assume that A is r_1 -connected and C is r_2 -connected. Then $A \rightarrow Y \rightarrow C$ is a homotopy fiber sequence in dimensions $\leq r_1 + r_2$.*

Proof. This is a direct application of Blakers-Massey excision theorem for commutative squares. \square

The following is now a straightforward consequence

Corollary 2.2. *Assume in addition that $Z \in \mathcal{T}(X)$ is an object with $\dim Z \leq r_1 + r_2$. Then the sequence of sets*

$$[Z, A]_{\mathcal{T}(X)} \rightarrow [Z, Y]_{\mathcal{T}(X)} \rightarrow [Z, C]_{\mathcal{T}(X)}$$

is exact.

(Explanation: the set $[Z, C]_{\mathcal{T}(X)}$ has a preferred basepoint given by $Z \rightarrow X' \rightarrow C$. Any element of $[Z, Y]_{\mathcal{T}(X)}$ which maps to the basepoint lifts back to $[Z, A]_{\mathcal{T}(X)}$.)

Fiberwise suspension. The *unreduced fiberwise suspension* of an object $Y \in \mathcal{T}(X)$ is the object of $\mathcal{R}(X)$ given by the double mapping cylinder

$$S_X Y := X \times 0 \cup Y \times [0, 1] \cup X \times 1$$

where the structure map $S_X Y \rightarrow X$ is obvious and the structure map $X \rightarrow S_X Y$ is given by $X \times 0$. This gives a functor $S_X : \mathcal{T}(X) \rightarrow \mathcal{R}(X)$. Similarly, $\mathcal{R}(X)$ has an *reduced* suspension functor $\Sigma_X : \mathcal{R}(X) \rightarrow \mathcal{R}(X)$ defined as follows: given an object $Y \in \mathcal{R}(X)$, we take $\Sigma_X Y$ to be the pushout of the diagram $X \leftarrow S_X X \rightarrow S_X Y$. If Y is cofibrant, then the map $S_X Y \rightarrow \Sigma_X Y$ is a weak equivalence. The functor Σ_X has a right adjoint Ω_X , called the *fiberwise loop functor*.

Let $X \times S^0 \rightarrow X$ be the first factor projection and let $\mathcal{T}(X \times S^0 \rightarrow X)$ be the category of spaces Y equipped with factorization $X \times S^0 \rightarrow Y \rightarrow X$. A morphism is a map of spaces which is compatible with the factorization data. Then unreduced suspension is possibly better regarded as a functor $S_X : \mathcal{T}(X) \rightarrow \mathcal{T}(X \times S^0 \rightarrow X)$. With this convention, S_X admits a right adjoint

$$O_X : \mathcal{T}(X \times S^0 \rightarrow X) \rightarrow \mathcal{T}(X).$$

Given $Y \in \mathcal{T}(X \times S^0 \rightarrow X)$, let $-, + : X \rightarrow Y$ denote the two inclusions determined by the structure map $X \times S^0 \rightarrow Y$. Then $O_X Y$ is the space consisting of triples (x_1, λ, x_2) in which $x_1, x_2 \in X$ and $\lambda : [0, 1] \rightarrow Y$ is a path such that $-(x_1) = \lambda(0)$ and $+(x_2) = \lambda(1)$. For example, when $X = x$ is a point and Y is a space with two distinct basepoints $a, b \in Y$, then $O_x Y$ is the space of paths in Y from a to b .

Given objects $Y, Z \in \mathcal{R}(X)$ define

$$\{Y, Z\}_{\mathcal{R}(X)} := \operatorname{colim}_k [\Sigma_X^k Y, \Sigma_X^k Z].$$

This is the abelian group of fiberwise stable homotopy classes from Y to Z .

Fiberwise smash product. Given objects $Y, Z \in \mathcal{T}(X)$, we have the fiber product $Y \times_X Z \in \mathcal{T}(X)$ which is given as the limit of the diagram $Y \rightarrow X \leftarrow Z$. If $Y, Z \in \mathcal{R}(X)$, the (fiberwise) wedge $Y \vee_X Z$ is the object of $\mathcal{R}(X)$ given by the pushout of the inclusions $Y \supset X \subset Z$. The (*internal fiberwise*) *smash product* is the object $Y \wedge_X Z$ given by the pushout of the diagram $X \leftarrow Y \vee_X Z \subset Y \times_X Z$. As is usual with most functors in the model category theoretic setting, this construction needs to be suitably derived to have a meaningful homotopy type. To avoid notational clutter, we will be intentionally sloppy: we will write the underived smash product but the reader should understand that it needs to be derived to have a sensible homotopy theoretic meaning.

There is also an *external* version of the above. Suppose That $A \in \mathcal{R}(X)$ and $B \in \mathcal{R}(Y)$. Then the colimit of the diagram

$$X \times X \leftarrow A \times X \cup X \times B \rightarrow A \times B$$

is denoted by $A_X \wedge_Y B$. It is an object of $\mathcal{R}(X \times Y)$. The association gives rise to a functor $\mathcal{R}(X) \times \mathcal{R}(Y) \rightarrow \mathcal{R}(X \times Y)$. Note that

$$(A_X \wedge_Y B)/(X \times Y) = (A/X) \wedge (B/Y).$$

In the special case when $X = Y$ we have $\Delta^*(A_X \wedge_X B) = A \wedge_X B$. Here $\Delta^*: \mathcal{R}(X \times X) \rightarrow \mathcal{R}(X)$ is the given by taking the pullback along the diagonal map $\Delta: X \rightarrow X \times X$.

Fiberwise Thom spaces. Given an object $Y \in \mathcal{T}(X)$ and an inner product bundle ξ over Y , the *fiberwise Thom space* is the object of $\mathcal{R}(X)$ given by

$$T_X(\xi) = D(\xi) \cup_{S(\xi)} X.$$

By collapsing X to a point we obtain usual Thom space of $D(\xi)/S(\xi)$, which in the present notation appears as $T_*(\xi)$.

If η is an inner product bundle over another object $Z \in \mathcal{T}(X)$. Then the fiber product $Y \times_X Z$ is again an object of $\mathcal{T}(X)$. Let $p: Y \times_X Z \rightarrow Y$ and $q: Y \times_X Z \rightarrow Z$ be the projections. Then the Whitney sum $p_*\xi \oplus q_*\eta$ is an inner product bundle over $Y \times_X Z$. The following is just an unravelling of definitions (and is well-known when X is a point). We leave its proof to the reader.

Lemma 2.3. *There is a preferred homeomorphism*

$$T_X(p_*\xi \oplus q_*\eta) \cong T_X(\xi) \wedge_X T_X(\eta).$$

Fiberwise spectra. Using Σ_X also enables one define spectra built from objects of $\mathcal{R}(X)$. These are sometimes called or *fiberwise spectra* or *parametrized spectra* (we will stick to the first name for these). A fiberwise spectrum \mathcal{E} is a collection of objects $\mathcal{E}_n \in \mathcal{R}(X)$ for $n = 0, 1, \dots$ together with morphisms $\Sigma_X \mathcal{E}_n \rightarrow \mathcal{E}_{n+1}$. A morphism of fiberwise spectra is the evident thing. Fiberwise spectra form a model category (see e.g., [Sc]).

Here are two examples:

Example 2.4 (Trivial fiberwise spectra). Starts with an ordinary spectrum E given by based spaces $\{E_n\}_{n \geq 0}$ and structure maps $\Sigma E_n \rightarrow E_{n+1}$. Form $E_n \times X$ for $n \geq 0$. These fit into a fiberwise spectrum $E \times X$, where the structure map $\Sigma_X(E_n \times X) \rightarrow E_{n+1} \times X$ is given by noticing that $\Sigma_X(E_n \times X) = (\Sigma E_n) \times X$.

Example 2.5 (Fiberwise suspension spectra). Start with any object $Y \in \mathcal{R}(X)$ and form the iterates $\Sigma_X^n Y$. These give a fiberwise spectrum $\Sigma_X^\infty Y$, using the identity maps for the structure maps.

Given an object $Z \in \mathcal{R}(X)$ and a fiberwise spectrum \mathcal{E} we define

$$\{Z, \mathcal{E}\}_{\mathcal{R}(X)} := \operatorname{colim}_n [\Sigma_X Z, \mathcal{E}_{k+n}]_{\mathcal{R}(X)}.$$

(This is slight abuse of notation: for example if $\mathcal{E} = \Sigma_X^\infty Y$ is a fiberwise suspension spectrum, we recover $\{Z, Y\}_{\mathcal{R}(X)}$.)

Homology and cohomology. Let \mathcal{E} be a fiberwise spectrum over X . Then an object $Z \in \mathcal{T}(X)$ (which we assume to be cofibrant) with structure map $p: Z \rightarrow X$ gives rise to a fiberwise spectrum over Z $p^* \mathcal{E}$ whose k -th term is the pullback of $\mathcal{E}_k \rightarrow X$ along p . This is not in general a homotopy invariant construction; to make it homotopy invariant we derive it by first performing a fibrant replacement \mathcal{E}^\sharp of \mathcal{E} . Let $(p^* \mathcal{E}^\sharp)^\flat$ denote the effect of making $p^* \mathcal{E}^\sharp$ cofibrant. Then for each $n \geq 0$ we have a cofibration $Z \rightarrow (p^* \mathcal{E}^\sharp)_n^\flat$ and as n varies the quotient spaces $(p^* \mathcal{E}^\sharp)_n^\flat / Z$ forms a spectrum denoted $H_\bullet(Z; \mathcal{E})$. The homology groups of Z with coefficients in \mathcal{E} are the homotopy groups of this spectrum.

Cohomology is somewhat easier to define. Make \mathcal{E} fibrant to obtain \mathcal{E}^\sharp . Then take the space of sections of $\mathcal{E}_n^\sharp \rightarrow X$ along the map $Z \rightarrow X$ (this is the same thing as the space of maps $Z \rightarrow \mathcal{E}_n$ which commute with the structure map to X). As n -varies, these spaces form a spectrum $H^\bullet(Z; \mathcal{E})$. The cohomology groups of Z with coefficients in \mathcal{E} are defined to be homotopy groups of this spectrum. Note that $H^0(Z; \mathcal{E}) = \{Z^+, \mathcal{E}\}_{\mathcal{R}(X)}$.

2.1. Induction and restriction. Let $f: X \rightarrow Y$ be a map of spaces. Then a fiberwise spectrum \mathcal{E} over Y gives rise to a fiberwise spectrum $f^*\mathcal{E}$ over X by taking base change. This operation defines a functor, called *restriction* from fiberwise spectra over Y to fiberwise spectra over X . The construction is homotopy invariant provided \mathcal{E} is fibrant. Using f to regard X as an object of $\mathcal{T}(Y)$, we obtain a tautological identification $H^\bullet(X; \mathcal{E}) = H^\bullet(X, f^*\mathcal{E})$, where on the right side X is viewed as an object of $\mathcal{R}(X)$ using the identity.

Suppose \mathcal{F} is a fiberwise spectrum over X . Then we obtain a fiberwise spectrum over Y , denoted $f_*\mathcal{F}$ in which $(f_*\mathcal{F})_k = (\mathcal{F}_k) \cup_f Y$. The construction is homotopy invariant when \mathcal{F} is cofibrant. Then $H_\bullet(X; \mathcal{F}) = H_\bullet(Y; f_*\mathcal{F})$ tautologically.

Poincaré duality. ξ be a finite dimensional vector bundle over X . let S^ξ be the fiberwise one-point compactification of ξ . Then S^ξ is an object of $\mathcal{R}(X)$. More generally, if ξ is a virtual bundle, i.e., $\xi + \epsilon^j$ is identified with a finite dimensional vector bundle η , then we define S^ξ in this case to be a fiberwise spectrum over X given by the j -fold desuspension of S^η .

Given a fiberwise spectrum \mathcal{E} over X , set

$${}^\xi\mathcal{E} := S^\xi \wedge_X \mathcal{E}.$$

When ξ is a vector bundle then the definition of the right side is given by the fiberwise smash products in each degree, i.e., $S^\xi \wedge_X \mathcal{E}_k$. In virtual bundle case one merely fiberwise desuspends $S^\eta \wedge_X \mathcal{E}$ j -times.

Theorem 2.6 (Poincaré duality). *Suppose $f: P \rightarrow X$ is a map in which P is a closed smooth manifold of dimension d . Let $-\tau_P$ be the virtual bundle given by the negation of the tangent bundle of P . Then for any fiberwise spectrum \mathcal{E} over X there is a weak equivalence of spectra*

$$H^\bullet(P; \mathcal{E}) \simeq H_\bullet(P; {}^{-\tau_P}f^*\mathcal{E}).$$

Bordism. We review the definition of bordism with coefficients in a virtual bundle. Let X be a space equipped with a finite dimensional inner product bundle ξ . Then one has the Thom space X^ξ which is quotient space formed from the unit disk bundle by collapsing the unit sphere bundle to a point. For the purposes of this paper, we define $\Omega_k(X; \xi)$ to be the k -th stable homotopy group of X^ξ . By standard transversality arguments, an element of this abelian group is generated by a compact smooth submanifold $V \subset \mathbb{R}^{k+d}$, for some $d \geq 0$, together with a map $g: V \rightarrow X$ such that the pullback along g of $\xi \oplus \epsilon^d$ is identified with the normal bundle of V , (where ϵ^d is the trivial bundle of rank

d). The relation of bordism of such data defines an equivalence relation and the set of equivalence classes is identified with $\Omega_k(X; \xi)$. With respect to this identification, the operation of disjoint union corresponds to the additive structure on $\pi_k^{st}(X^\xi)$. Now suppose that ξ is a virtual bundle. This means that $\xi \oplus \epsilon^j$ comes equipped with an isomorphism to a finite dimensional inner product bundle η , for some integer $j \geq 0$. In this instance, we define $\Omega_k(X; \xi)$ to be $\Omega_{k+j}(X; \eta)$. Our indexing convention for the bordism group differs from the one used in [KW1], but is identical to the one used in [KW2].

3. STRONGLY COCARTESIAN CUBES

For a set T let 2^T be the poset of consisting of the subsets of T partially ordered by inclusion. A $|T|$ -cube of in a category \mathcal{C} is a contravariant functor $A_\bullet : 2^T \rightarrow \mathcal{C}$. The notation is such that A_S denotes the value of the functor at S . Because we choose to work contravariantly, the initial vertex of A_\bullet is A_T and the terminal vertex is A_\emptyset . When $T = \{i\}$ we often write $A_i = A_T$.

For subsets $U \subset W \subset \mathcal{T}$, one has a $(|W - U|)$ -face of A_\bullet given by restricting A_\bullet to those A_W for which $U \subset V \subset W$. This is a $(|W - U|)$ -cube.

From now on we will only be considering j -cubes given by functors $2^{\mathcal{J}} \rightarrow \mathcal{C}$ where \mathcal{C} is one of the categories \mathcal{T} , $\mathcal{T}(X)$ or $\mathcal{R}(X)$. A *weak equivalence* of such j -cubes is a natural transformation $A_\bullet \rightarrow B_\bullet$ such that $A_T \rightarrow B_T$ is a weak equivalence for each T . Two cubes are said to be weak equivalent if there is a finite chain of weak equivalences connecting them.

Definition 3.1. A j -cube A_\bullet is said to be *strongly cocartesian* if each two dimensional face of A_\bullet is ∞ -cocartesian.

In the above definition, it is enough to check the condition on each 2-face meeting the initial vertex.

Example 3.2. Let X_1, \dots, X_j be cofibrant based spaces. Set $A_T = \bigvee_{i \in T} X_i$ if $T \neq \emptyset$ and let A_\emptyset be a point. The maps of the cube are given by projections onto summands. Then A_\bullet is strongly cocartesian.

More generally, let $X_1, X_j \in \mathcal{R}(X)$ be cofibrant retractive spaces over some space X . Let A_T be the fiberwise wedge of X_i as i varies in T . Then A_\bullet is strongly cocartesian. We say in this case that A_\bullet is a *wedge cube*.

Example 3.3. With $X_1, \dots, X_j \in \mathcal{R}(X)$ as above, let B_T be the fiberwise wedge of X_i where now i varies in $\mathcal{J} \setminus T$. The maps of this cube

are inclusions of summands. Then B_\bullet is strongly cocartesian. We say that B_\bullet is a *backwards wedge cube*.

Example 3.4. Let A_\bullet be a strongly cocartesian cube of $\mathcal{T}(X)$, then the cube $S_X A_\bullet$ given by taking the unreduced fiberwise suspension of each A_T , is strongly cocartesian. Similarly, if A_\bullet is strongly cocartesian cube of $\mathcal{R}(X)$ then so is $\Sigma_X A_\bullet$.

Lemma 3.5. *If A_\bullet is a strongly cocartesian j -cube of in $\mathcal{R}(X)$ in which $X = A_\emptyset$. Assume $A_T \in \mathcal{R}(X)$ is cofibrant for all T . Then the j -cube $\Sigma_X A_\bullet$ is weak equivalent to a wedge cube B_\bullet in which $B_i = \Sigma_X A_i$ for all $i \in \mathcal{J}$.*

Proof. The following argument was explained to us by Tom Goodwillie. First consider the case when X is a point. Let B_T be the wedge of ΣA_i for all $i \in T$, but write this as the wedge, over all $i \in \mathcal{J}$, of either

- ΣA_i if $i \in T$, or
- a point if $i \notin T$.

Define a map $\Sigma A_T \rightarrow B_T$ as follows. First do a pinch to go from ΣA_T to the wedge of j copies of ΣA_T indexed by $i \in \mathcal{J}$. Now map that to B_T by sending the i -th copy of ΣA_T to ΣA_i using the original map $A_T \rightarrow A_i$ if $i \in T$, or the constant map to point if $i \notin T$.

This defines a map $\Sigma A_\bullet \rightarrow B_\bullet$ which has the property that when $T = \{i\}$ the map from $\Sigma A_T = \Sigma A_i$ to $B_T = \Sigma A_i$ is homotopic to the identity. Hence the map of cubes is a weak equivalence. This takes care of the case when X is a point. The general case is along the same lines (we leave the details to the reader). \square

Corollary 3.6. *Suppose A_\bullet is a strongly cocartesian j -cube of $\mathcal{T}(X)$, where $X = A_\emptyset$. Then the j -cube $S_X^2 A_\bullet$ is weak equivalent to a wedge cube B_\bullet in which $B_i = S_X^2 A_i$ for $i \in \mathcal{J}$.*

Proof. The natural map $S_X^2 A_\bullet \rightarrow \Sigma_X S_X A_\bullet$ is a weak equivalence. The result follows by applying Lemma 3.5 to $S_X A_\bullet$ (considered as a cube of $\mathcal{R}(X)$). \square

Given a strongly cocartesian j -cube A_\bullet , let $C(A_\bullet)$ denote the homotopy colimit

$$\mathrm{hocolim}(A_\emptyset \leftarrow A_{\mathcal{J}} \rightarrow \mathrm{holim}_{S \neq \mathcal{J}} A_S).$$

Then $C(A_\bullet)$ is a retractive space over A_\emptyset . In what follows we rename

$$X := A_\emptyset.$$

Then $C(A_\bullet) \in \mathcal{R}(X)$ and one has a homotopy cofiber sequence

$$A_{\mathcal{J}} \rightarrow \operatorname{holim}_{S \neq \mathcal{J}} A_S \rightarrow C(A_\bullet).$$

Notation 3.7. (1). For a sequence of integers r_1, \dots, r_j we write

$$\Sigma = \sum_i r_i \quad \text{and} \quad \mu = \min_i r_i.$$

(2). If $1 \leq i \leq j$ and $T \subset \mathcal{J}$ set $T_i = T \setminus \{i\}$.

We make the following assumptions throughout.

Hypothesis 3.8. (1). The map

$$A_{\mathcal{J}} \rightarrow A_{\mathcal{J}_i}$$

is $(r_i + 1)$ -connected for $1 \leq i \leq j$. It follows that $A_T \rightarrow A_{T_i}$ is also $(r_i + 1)$ -connected, since A_\bullet is strongly cocartesian. If $i \in T$, the map is the identity, so it is ∞ -cartesian.

(2). The space $X = A_\emptyset$ is path connected.

Proposition 3.9. *Let $Z \in \mathcal{T}(X)$ be an object of dimension $\leq 1 + \mu + \Sigma$. Then the sequence*

$$[Z, A_{\mathcal{J}}]_{\mathcal{T}(X)} \rightarrow [Z, \operatorname{holim}_{S \neq \emptyset} A_S]_{\mathcal{T}(X)} \rightarrow [Z, C(A_\bullet)]_{\mathcal{T}(X)}$$

is exact.

Proof. The generalized Blakers-Massey theorem for cubical diagrams. \square

Identification of $C(A_\bullet)$. In the remainder of this section we identify $C(A_\bullet)$ in dimensions $\leq 2 + \mu + \Sigma$.

Let $S_X A_\bullet$ be the strongly cocartesian cube obtained from A_\bullet given by $T \mapsto S_X A_T$.

Theorem 3.10. *There is a morphism of $\mathcal{R}(X)$*

$$S_X^j C(A_\bullet) \rightarrow C(S_X A_\bullet)$$

which is $(2 + j + \Sigma + \mu)$ -connected.

Remark 3.11. Modulo the difference between reduced and unreduced fiberwise suspension, we infer from 3.10 that the collection

$$\mathcal{C}(A_\bullet) := \{C(S_X^k A_\bullet)\}_{k \geq 0}$$

forms a fiberwise spectrum over X where the structure maps involve j -fold rather than one-fold suspension. In §5, we will fill in the missing spaces and maps to obtain a fiberwise spectrum over X whose structure maps stabilize one dimension at a time.

Corollary 3.12. *Suppose that $Z \in \mathcal{R}(X)$. Then the map*

$$[Z, C(A_\bullet)]_{\mathcal{R}(X)} \rightarrow \{Z, \mathcal{C}(A_\bullet)\}_{\mathcal{R}(X)}$$

is a surjective if $\dim Z \leq 2 + \mu + \Sigma$, and a bijection if $\dim Z \leq 1 + \mu + \Sigma$.

Let $W_j = \bigvee_{(j-1)!} S^{2-2j}$ be the wedge of $(j-1)!$ -copies of the $(2-2j)$ -sphere spectrum. Let

$$\mathcal{W}_j = X \times W_j$$

be the trivial fiberwise spectrum on W_j .

Theorem 3.13. *There is a (weak) map of fiberwise spectra*

$$\mathcal{W}_j \wedge_X \left(\bigwedge_{i \in \mathcal{J}} S_X A_i \right) \rightarrow \mathcal{C}(A_\bullet)$$

which is $(4 + j + \Sigma + \mu)$ -connected.

(A weak map from A to B is by definition a diagram $A \rightarrow B' \leftarrow B$ in which $B \rightarrow B'$ is a weak equivalence). The proof of 3.13 is deferred to §6.

Corollary 3.14. *Let $Z \in \mathcal{T}(X)$ be an object such that $\dim Z \leq 1 + \mu + \Sigma$. Then there is an exact sequence*

$$[Z, A_{\mathcal{J}}]_{\mathcal{T}(X)} \rightarrow [Z, \operatorname{holim}_{S \neq \emptyset} A_S]_{\mathcal{T}(X)} \rightarrow \{Z^+, \mathcal{W}_j \wedge_X \left(\bigwedge_{i \in \mathcal{J}} S_X A_i \right)\}_{\mathcal{R}(X)}.$$

Remark 3.15. Corollary 3.14 is a fiberwise version of a result of Barratt and Whitehead [BW].

The Euler class. Let $f: Z \rightarrow \operatorname{holim}_{S \neq \emptyset} A_S$ be a map of spaces. Then Z is automatically a morphism of $\mathcal{T}(X)$. Using the identifications of 3.12 and 3.13, we see that the composed map

$$Z^+ \xrightarrow{f} \operatorname{holim}_{S \neq \emptyset} A_S \rightarrow C(A_\bullet)$$

gives rise to a fiberwise stable homotopy class

$$e(f) \in \{Z^+, \mathcal{W}_j \wedge_X \left(\bigwedge_{i \in \mathcal{J}} S_X A_i \right)\}_{\mathcal{R}(X)}$$

which we call the *Euler class* of f . Equivalently, $e(f)$ resides in the cohomology group

$$H^0(Z; \mathcal{W}_j \wedge_X \left(\bigwedge_{i \in \mathcal{J}} S_X A_i \right)).$$

Then from 3.14 we immediately deduce

Corollary 3.16. *The Euler class $e(f)$ vanishes when f admits a homotopy factorization through $A_{\mathcal{J}}$. Conversely, when $\dim Z \leq 1 + \mu + \Sigma$ and $e(f) = 0$, then f admits a homotopy factorization through $A_{\mathcal{J}}$.*

4. THE INTERSECTION THEORY CASE

Let $N - Q_\bullet$ be the cubical diagram given by $\{N - Q_S\}_{S \subset \mathcal{J}}$. Recall that we are given a map $f: P \rightarrow \operatorname{holim}_{S \neq \mathcal{J}} (N - Q_S)$ and we wish to identify the obstructions to deforming it into $N - Q_{\mathcal{J}}$.

By transversality, the map $N - Q_{\mathcal{J}} \rightarrow N - Q_{\mathcal{J} - \{i\}}$ is $n - q_i - 1$ connected for $1 \leq i \leq j$. Using 3.16, we deduce

Proposition 4.1. *If $P \rightarrow \operatorname{holim}_{S \neq \mathcal{J}} N - Q_S$ admits a homotopy factorization through $N - Q_{\mathcal{J}}$, then $e(f) = 0$. The converse is true provided $p \leq 1 + \mu + \Sigma$, where $\Sigma = \sum_i (n - q_i - 2)$ and $\mu_i = \min_i (n - q_i - 2)$.*

(Here we are using the fact that the closed manifold P of dimension p admits a cell structure with cells in dimensions $\leq p$.)

Let ν_i be the normal bundle of Q_i in N . By the tubular neighborhood theorem there is a weak equivalence of retractive spaces

$$S_N(N - Q_1) \simeq D(\nu_i) \cup_{S(\nu_i)} N$$

where the right side is the fiberwise Thom space of ν over N , which we will denote by $T_N(\nu_{Q_i})$. It is an object of $\mathcal{R}(X)$.

Stably, we can identify ν_i with the virtual bundle $\xi_i := f^* \tau_N - \tau_{Q_i}$, given by the difference of tangent bundles. We write $T_N(\xi_i)$ for the associated fiberwise Thom spectrum. With these notational changes, $e(f)$ can be regarded as residing in the cohomology group

$$H^0(P; \mathcal{W}_j \wedge_N (\bigwedge_{i \in \mathcal{J}} T_N(\xi_i))).$$

The Euler characteristic. By the Poincaré duality isomorphism, $e(f)$ corresponds to a homology class

$$\chi(f) \in H_0(P; {}^{-\tau_P} f^* (\mathcal{W}_j \wedge_N (\bigwedge_{i \in \mathcal{J}} T_N(\xi_i)))).$$

Using the induction isomorphism, the group where $\chi(f)$ resides can alternatively be written as

$$H_0(N; f_* {}^{-\tau_P} f^* (\mathcal{W}_j \wedge_N (\bigwedge_{i \in \mathcal{J}} T_N(\xi_i)))).$$

By definition, the latter is the stable homotopy group in degree zero of the ordinary spectrum

$$(\mathcal{W}_j \wedge_N T_N(-\tau_P) \wedge_N (\bigwedge_{i \in \mathcal{J}} T_N(\xi_i))) / N.$$

Using 2.3, or rather a virtualized version of it, we immediately see that the fiberwise spectrum

$$\mathcal{W}_j \wedge_N T_N(-\tau_P) \wedge_N (\bigwedge_{i \in \mathcal{J}} T_N(\xi_i))$$

can be rewritten up to homotopy as

$$\mathcal{W}_j \wedge_N T_N(\xi)$$

where ξ is the virtual inner product bundle of $E(P, Q_\bullet)$ that was defined in the introduction. Now \mathcal{W}_j is just the fiberwise wedge of $(j-1)!$ copies of the fiberwise spectrum $N \times S^{2-2j}$. From this we easily deduce

$$(\mathcal{W}_j \wedge_N T_N(\xi))/N \simeq \vee_{(j-1)!} \Sigma^{2-2j} E(P, Q_\bullet)^\xi.$$

Since

$$\pi_0(\Sigma^{2-2j} E(P, Q_\bullet)^\xi) \cong \Omega_{2j-2}(E(P, Q_\bullet); \xi)$$

we conclude

Theorem 4.2. *The obstruction $\chi(f)$ resides in the abelian group*

$$\bigoplus_{(j-1)!} \Omega_{2j-2}(E(P, Q_\bullet); \xi)$$

It vanishes whenever $f: P \rightarrow \text{holim}_{S \neq \mathcal{J}} (N - Q_S)$ admits a homotopy factorization through $N - Q_{\mathcal{J}}$. Conversely, if $p \leq 1 + \mu + \Sigma$, then $\chi(f) = 0$ implies there is such a factorization of f .

5. PROOF OF THEOREM 3.10

We first construct the map

$$S_X^j C(A_\bullet) \rightarrow C(S_X A_\bullet).$$

The idea goes like this: for $i = 1, \dots, j$ we will define an endo-functor f_i of strongly cocartesian j -cubes with the property that $f_j \circ f_{j-1} \circ \dots \circ f_1(A_\bullet)$ is identified up to weak equivalence with the fiberwise suspension $S_X A_\bullet$. Furthermore, we will construct a map

$$S_X C(A_\bullet) \rightarrow C(f_i A_\bullet).$$

for each i . The desired map will be the composition of the latter maps as i is iterated from 1 to j .

Notation 5.1. For $T \subset U \subset \mathcal{J}$, set

$$S_T A_U := S_{A_T} A_U$$

be the unreduced fiberwise suspension of A_U over A_T .

We proceed to define the functor f_i . If $T \subset \mathcal{J}$ recall the notation $T_i = T \setminus \{i\}$, and define $f_i A_T = S_{T_i} A_T$. Then the collection $\{f_i A_T\}_{T \subset \mathcal{J}}$ defines $f_i A_\bullet$. We leave it to the reader to check that this cube is strongly cocartesian. Let $g_i f_i A_T = O_{T_i} S_{T_i} A_T$. Then there is a natural map of j -cubes

$$A_\bullet \rightarrow g_i f_i A_\bullet.$$

which induces a commutative square

$$(1) \quad \begin{array}{ccc} A_{\mathcal{J}} & \longrightarrow & O_{\mathcal{J}_i} S_{\mathcal{J}_i} A_{\mathcal{J}} \\ \downarrow & & \downarrow \\ \operatorname{holim}_{T \neq \mathcal{J}} A_T & \longrightarrow & \operatorname{holim}_{T \neq \mathcal{J}} O_{T_i} S_{T_i} A_T. \end{array}$$

We will need to exhibit a commutation rule for the homotopy limit appearing in the lower right corner.

Lemma 5.2. *For $U, T, V \subset \mathcal{J}$ with $T \supset V \subset U$, there is a preferred weak equivalence*

$$\operatorname{holim}(A_T \rightarrow A_V \leftarrow O_V S_V A_U) \simeq O_T \operatorname{holim}(A_T \rightarrow A_V \leftarrow S_V A_U).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} A_T & \longrightarrow & A_V & \xlongequal{\quad} & A_V \\ \parallel & & \parallel & & \downarrow - \\ A_T & \longrightarrow & A_V & \longleftarrow & S_V A_U \\ \parallel & & \parallel & & \uparrow + \\ A_T & \longrightarrow & A_V & \xlongequal{\quad} & A_V, \end{array}$$

where the maps labelled “+,” “−” are the canonical sections of the fiberwise suspension. If we take the homotopy limit of each row and thereafter the homotopy limit of the resulting diagram, we obtain the right side of 5.2. Alternatively, if we take the homotopy limit of each column and then the homotopy limit of the resulting diagram, we obtain the left side of 5.2. The conclusion therefore follows from the commutativity of homotopy limits. \square

Corollary 5.3. *There is a preferred weak equivalence*

$$O_{\mathcal{J}_i} \operatorname{holim}_{T \neq \mathcal{J}} S_{T_i} A_T \simeq \operatorname{holim}_{T \neq \mathcal{J}} O_{T_i} S_{T_i} A_T.$$

Inserting 5.3 into the lower right-hand corner of the square (1), we obtain

$$(2) \quad \begin{array}{ccc} A_{\mathcal{J}} & \longrightarrow & O_{\mathcal{J}_i} S_{\mathcal{J}_i} A_{\mathcal{J}} \\ \downarrow & & \downarrow \\ \operatorname{holim}_{T \neq \mathcal{J}} A_T & \longrightarrow & O_{\mathcal{J}_i} \operatorname{holim}_{T \neq \mathcal{J}} S_{T_i} A_T \end{array}$$

where the right vertical map arises by applying $O_{\mathcal{J}_i}$ to the canonical map associated with the cube $f_1 A_{\bullet}$.

By taking adjoints, we obtain another commutative square

$$(3) \quad \begin{array}{ccc} S_{\mathcal{J}_i} A_{\mathcal{J}} & \xlongequal{\quad} & S_{\mathcal{J}_i} A_{\mathcal{J}} \\ \downarrow & & \downarrow \\ S_{\mathcal{J}_i} \operatorname{holim}_{T \neq \mathcal{J}} A_T & \longrightarrow & \operatorname{holim}_{T \neq \mathcal{J}} S_{\mathcal{T}_i} A_T . \end{array}$$

Taking homotopy cofibers of the latter (in $\mathcal{T}(X)$) vertically, we then obtain the desired morphism of $\mathcal{R}(X)$

$$S_X C(A_{\bullet}) \rightarrow C(f_1 A_{\bullet}) .$$

We next need to show that the iterated composition $f_j \cdots f_1 A_{\bullet}$ is identified with $S_X A_{\bullet}$. We will illustrate the proof when $j = 2$ and explain in slightly less detail the remaining cases to the reader.

When $j = 2$, the goal is to identify $f_2 \circ f_1 A_{\bullet}$, which is the ∞ -cocartesian square

$$\begin{array}{ccc} S_2 S_1 A_{12} & \longrightarrow & S_X A_2 \\ \downarrow & & \downarrow \\ S_X A_1 & \longrightarrow & X \end{array}$$

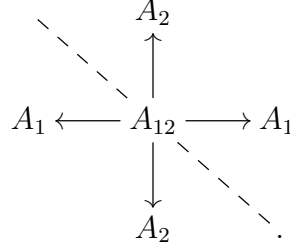
whose initial vertex can be identified with the homotopy colimit of the diagram

$$\begin{array}{ccccc} X & \longleftarrow & A_2 & \longrightarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ A_1 & \longleftarrow & A_{12} & \longrightarrow & A_1 \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & A_2 & \longrightarrow & X . \end{array}$$

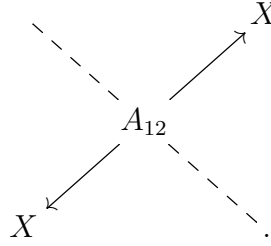
As each of the four squares of this diagram is ∞ -cocartesian, the homotopy colimit coincides up to homotopy with that of the subdiagram

$$\begin{array}{ccccc} & & A_2 & & \\ & & \uparrow & & \\ A_1 & \longleftarrow & A_{12} & \longrightarrow & A_1 \\ & & \downarrow & & \\ & & A_2 & & . \end{array}$$

The homotopy colimit of the latter can now be identified in two stages. First bisect the diagram into two pieces:



Next compute the homotopy colimit of each side of the dotted line to give:



Finally, compute the homotopy colimit of what remains, to get $S_X A_{12}$. This completes the $j = 2$ case.

When $j > 2$, the proof is analogous. In this instance we need to identify $f_j \cdots f_1 A_T$ with $S_X A_T$ for each $T \subset \mathcal{J}$. An unravelling of the definition shows that $f_j \cdots f_1 A_T$ is the homotopy colimit of a certain diagram indexed by the *cubical subdivision* of 2^T . The cubical subdivision is given by the partially ordered set \mathcal{P}_T whose elements consist of all faces of 2^T . There are $3^{|T|}$ of these, indexed by pairs (U, W) in which $U \subset W \subset T$. The order relation in \mathcal{P}_T is given by inclusion of faces. Define a contravariant functor $B : \mathcal{P}_T \rightarrow \mathcal{T}$ as follows: for an element $(U, W) \in \mathcal{P}_T$, we set $B(U, W) = A_{W-U}$. Then the homotopy colimit of B is identified with $f_j \cdots f_1 A_T$.

It suffices to identify the homotopy colimit of B with $S_X A_T$. As in the $j = 2$ case, we can restrict ourselves to the sub-poset of \mathcal{P}_T given by those (U, W) in which $W = T$ and $|U| \leq 1$, and then compute the homotopy colimit of the restriction. The reason we can make such a simplification is that B is obtained by gluing together $2^{|T|}$ identical copies of the strongly cocartesian cube A_\bullet along codimension one faces. Notice that this restriction is given gluing together two copies of the diagram

$$\{A_T \rightarrow A_{T_i}\}_{1 \leq i \leq |T|}$$

along the initial vertex A_T . Then the homotopy colimit of the displayed diagram is of course just X , so when we glue the two copies together the

homotopy colimit of the glued diagram is identified with $\text{hocolim}(X \leftarrow A_T \rightarrow X)$, and the latter is $S_X A_T$.

We have now identified $f_j \cdots f_1 A_T$ as $S_X A_T$. We leave it to the reader to verify that as $T \subset \mathcal{J}$ is allowed to vary, the identifications match up, yielding an identification of j -cubes $f_j \cdots f_1 A_\bullet \simeq S_X A_\bullet$.

The final step of the proof of Theorem 3.10 is to compute the connectivity of the map $S_X^j C(A_\bullet) \rightarrow C(S_X A_\bullet)$. By construction, this map factors as

$$S_X^j C(A_\bullet) \rightarrow S_X^{j-1} C(f_1 A_\bullet) \rightarrow \cdots \rightarrow C(f_j \cdots f_1 A_\bullet) = C(S_X A_\bullet),$$

in which the k -th map

$$S_X^{j-k+1} C(f_{k-1} \cdots f_1 A_\bullet) \rightarrow S_X^{j-k} C(f_k \cdots f_1 A_\bullet)$$

is the $(j-k)$ -fold suspension of the map constructed above, as applied to the cube $f_{k-1} \cdots f_1 A_\bullet$. Consequently, it will be enough to show that the map

$$C(f_{k-1} \cdots f_1 A_\bullet) \rightarrow O_X C(f_k \cdots f_1 A_\bullet)$$

is $(1+k+\Sigma+\mu)$ -connected. In fact, it is really enough to consider just the case $k=1$, since the map for each k is given by the same kind of construction applied to the j -cube $f_{k-1} \cdots f_1 A_\bullet$ which has the property that the map $f_{k-1} \cdots f_1 A_{\mathcal{J}} \rightarrow f_{k-1} \cdots f_1 A_{\mathcal{J}_i}$ is r_i+2 -connected if $1 \leq i \leq k$ and r_i -connected if $k+1 \leq i \leq j$. We leave the verification of these assertions to the reader.

Assume from now on that $k=1$. We are then reduced to showing that the map $C(A_\bullet) \rightarrow O_X C(f_1 A_\bullet)$ is $(2+\Sigma+\mu)$ -connected. The proof of this will require some preparation.

Consider the 3-cube

$$(4) \quad \begin{array}{ccccc} A_{\mathcal{J}} & \xrightarrow{\quad} & O_{\mathcal{J}_1} S_{\mathcal{J}_1} A_{\mathcal{J}} & & \\ & \searrow & \downarrow & \searrow & \\ & \text{holim}_{T \neq \mathcal{J}} A_T & \xrightarrow{\quad} & O_{\mathcal{J}_1} \text{holim}_{T \neq \mathcal{J}} f_1 A_T & \\ & \downarrow & & \downarrow & \\ X & \xrightarrow{\quad} & X & & \\ & \searrow & \downarrow & \searrow & \\ & C(A_\bullet) & \xrightarrow{\quad} & O_X C(f_1 A_\bullet) & \end{array}$$

Proposition 5.4. *For the cube (4), the following assertions are valid.*

- (1). *The top face is $(1+2r_1+\Sigma)$ -cartesian.*
- (2). *The left and right face are $(1+\mu+\Sigma)$ -cartesian.*

Proof. To prove the first part consider the square of j -cubes

$$\begin{array}{ccc} A_{\bullet} & \longrightarrow & X_{\bullet} \\ \downarrow & & \downarrow \\ X_{\bullet} & \longrightarrow & f_1 A_{\bullet} \end{array}$$

in which $X_T = A_{T_i}$ for all $T \subset \mathcal{J}$. Then an unravelling of definitions shows that $(j+2)$ -cube associated to this square is k -cartesian if and only if the top face of the cube (4) is k -cartesian. It is easy to show that this $(j+2)$ -cube is strongly cartesian. The result now follows from the generalized Blakers-Massey theorem applied to the $(j+2)$ -cube. This completes the argument which proves the first part.

Consider next the left face of the 3-cube (4). This square is ∞ -cocartesian. The top horizontal arrow of the square is the map $A_{\mathcal{J}} \rightarrow \operatorname{holim}_{T \neq \mathcal{J}} A_T$. This map is k -connected if and only if the j -cube A_{\bullet} is k -cartesian. By the generalized Blakers-Massey Theorem A_{\bullet} is $(1+\Sigma)$ -cartesian. Furthermore, the left vertical map $A_{\mathcal{J}} \rightarrow X$ is $(\mu+1)$ -connected. Now apply the Blakers-Massey theorem to deduce that the square is $(1+\mu+\Sigma)$ -connected.

We now argue that the right face of the 3-cube is $(1+\mu+\Sigma)$ -connected. The idea here is that the strongly cocartesian square

$$\begin{array}{ccc} S_{\mathcal{J}_1} A_{\mathcal{J}} & \longrightarrow & \operatorname{holim}_{T \neq \mathcal{J}} f_1 A_T \\ \downarrow & & \downarrow \\ X & \longrightarrow & C(f_1 A_{\bullet}) \end{array}$$

has the property that the top horizontal arrow is $(2+\Sigma)$ -connected and the left vertical arrow is $(1+\mu')$ -connected, where $\mu' = 1 + \min(r_1 + 1, r_2, \dots, r_j)$. From the Blakers-Massey theorem we deduce that the square is $(2+\mu'+\Sigma)$ -cartesian. Let Y_{\bullet} denote the above square, let Z_{\bullet} denote the right face of the cube, and let B_{\bullet} denote the ∞ -cartesian square

$$\begin{array}{ccc} A_{\mathcal{J}_1} & \xlongequal{\quad} & A_{\mathcal{J}_1} \\ \downarrow & & \downarrow \\ X & \xlongequal{\quad} & X \end{array}$$

By construction the square of squares

$$\begin{array}{ccc} Y_{\bullet} & \longrightarrow & B_{\bullet} \\ \downarrow & & \downarrow \\ B_{\bullet} & \longrightarrow & Z_{\bullet} \end{array}$$

which, when considered as a 4-cube, is ∞ -cartesian. Since Z_\bullet is $(2 + \mu' + \Sigma)$ -cartesian, it follows that Y_\bullet is $(1 + \mu' + \Sigma)$ -cartesian. Hence it is $(1 + \mu + \Sigma)$ -cartesian, since $\mu \leq \mu'$. \square

Proof that $C(A_\bullet) \rightarrow O_X C(f_1 A_\bullet)$ is $(2 + \Sigma + \mu)$ -connected. Let P denote the effect of taking the homotopy limit of the diagram given by the left face of the 3-cube (4) with its initial vertex removed. Similarly, let P' be the homotopy limit of the diagram given by removing the initial vertex of the right face. Then the 3-cube is k -cartesian if and only if the square

$$\begin{array}{ccc} A_{\mathcal{J}} & \longrightarrow & P \\ \downarrow & & \downarrow \\ O_{\mathcal{J}_1} S_{\mathcal{J}_1} A_{\mathcal{J}} & \longrightarrow & P' \end{array}$$

is k -cartesian. Since the left and right faces of the 3-cube are $(1 + \mu + \Sigma)$ -cartesian, the horizontal arrows in this square are $(1 + \mu + \Sigma)$ -connected. From this, one deduces in a straightforward way that the square is $(\mu + \Sigma)$ -cartesian (we leave these details to the reader). Hence the 3-cube is $(\mu + \Sigma)$ -cartesian.

Let Q be the homotopy limit of the top face of the cube (4) with its initial vertex removed, and similarly, let Q' be the corresponding homotopy limit for the bottom face. Then the 3-cube is k -cartesian if and only if the square

$$\begin{array}{ccc} A_{\mathcal{J}} & \longrightarrow & Q \\ \downarrow & & \downarrow \\ X & \longrightarrow & Q' \end{array}$$

is k -cartesian. Hence the square is $(\mu + \Sigma)$ -cartesian. The top map of this square is $(1 + 2r_1 + \Sigma)$ -connected by 5.4. In particular, it is $(1 + \mu + \Sigma)$ -connected. As the map $X \rightarrow Q' \rightarrow C(A_\bullet)$ is $(1 + \Sigma)$ -connected, and the map $X \rightarrow Q'$ is 0-connected. Since X is 0-connected, we infer that Q' is also 0-connected. In particular, the map $Q \rightarrow Q'$ is 0-connected. We then deduce by Lemma 5.5 below that the map $X \rightarrow Q'$ is $(1 + \mu + \Sigma)$ -connected.

By definition of Q' , we have an ∞ -cartesian square

$$\begin{array}{ccc} Q' & \longrightarrow & X \\ \downarrow & & \downarrow \\ C(A_\bullet) & \longrightarrow & \Omega_X C(f_1 A_\bullet) \end{array}$$

where the top horizontal map is a retraction to the map $X \rightarrow Q'$. Consequently, the map $Q' \rightarrow X$ is $(2 + \mu + \Sigma)$ -connected. Since the map $X \rightarrow \Omega_X C(f_1 A_\bullet)$ is $(1 + \Sigma)$ -connected, it is 0-connected, so by Lemma 5.5 again we conclude that the map $C(A_\bullet) \rightarrow \Omega_X C(f_1 A_\bullet)$ is $(2 + \mu + \Sigma)$ -connected. \square

The following result was referred to in the above argument.

Lemma 5.5. *Suppose*

$$\begin{array}{ccc} X_{12} & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_\emptyset \end{array}$$

is a $(r - 1)$ -cartesian diagram of spaces and the map $X_{12} \rightarrow X_2$ is r -connected. Assume also that the map $X_2 \rightarrow X_\emptyset$ is 0-connected. Then the map $X_1 \rightarrow X_\emptyset$ is also r -connected.

Proof. It is enough to check that the homotopy fiber of $X_1 \rightarrow X_\emptyset$ is $(r - 1)$ -connected for every choice of basepoint in X_\emptyset . In fact, we only need to check this for one representative of each path component of X_\emptyset . As $X_2 \rightarrow X_\emptyset$ is 0-connected, we can choose each basepoint as an image of a basepoint of X_2 . If $x \in X_2$ let F be the homotopy fiber of $X_{12} \rightarrow X_2$ at x and let F' be the homotopy fiber of $X_1 \rightarrow X_\emptyset$ taken at the image of x in X_\emptyset . Then the requirement that the square be $(r - 1)$ -cartesian is equivalent to the statement that the map $F \rightarrow F'$ is $(r - 1)$ -connected for every x . Since F is $(r - 1)$ -connected, we deduce that F' is also $(r - 1)$ -connected. Hence the map $X_1 \rightarrow X_\emptyset$ is r -connected. \square

6. PROOF OF THEOREM 3.13

The Hilton-Milnor theorem. Given connected cofibrant based spaces X_1, \dots, X_j , The Hilton-Milnor Theorem [A, p. 131] describes a homotopy equivalence

$$\Omega(\bigvee_{i=1}^j \Sigma X_i) \simeq \prod_{i=1}^{\infty} \Omega \Sigma X_i.$$

On the right hand side, we are displaying a weak product, and the term X_i for $i > j$ has the form

$$X_1^{(n_1)} \wedge \dots \wedge X_j^{(n_j)},$$

where $Y^{(k)}$ for a based space Y denotes the smash product of k copies of Y . The number of terms of a given form is

$$\phi(n_1, \dots, n_j) = \frac{1}{n} \sum_{d|\delta} \frac{\mu(d)(n/d)!}{(n_1/d)!(n_2/d)! \cdots (n_j/d)!}$$

in which $n = \sum_i n_i$ and $\delta = \gcd(d_1, \dots, d_j)$. In particular when $n = j$ and $d_i = 1$ for all i , we see that the number of terms of the form $X_1 \wedge \cdots \wedge X_r$ is $(j-1)!$. The map of the Hilton-Milnor theorem which gives the homotopy equivalence given from right to left is given by certain elementary Whitehead products (cf. 6.2 below).

If X_i is r_i connected, then the inclusion map

$$\prod_{i=1}^j \Omega \Sigma X_i \times \prod_{(j-1)!} \Omega \Sigma(X_1 \wedge \cdots \wedge X_j) \xrightarrow{\subseteq} \prod_{i=1}^{\infty} \Omega \Sigma X_i$$

is $(j + \Sigma + \mu)$ -connected (where as usual $\Sigma = \sum_i r_i$ and $\mu = \min_i r_i$). Consider the j -cubical diagram $S \mapsto \Sigma X_S$, where $X_S = \vee_{i \in S} X_i$, where S ranges through subsets of \mathcal{J} . The maps in the cube are projections onto summands. Using the above observations, it is straightforward to deduce the following.

Proposition 6.1. *There is a homotopy fiber sequence in dimensions $\leq 1 + j + \Sigma + \mu$*

$$\bigvee_{(j-1)!} \Sigma(X_1 \wedge \cdots \wedge X_j) \rightarrow \bigvee_{i=1}^j \Sigma X_i \rightarrow \operatorname{holim}_{S \neq \mathcal{J}} \left(\bigvee_S \Sigma X_S \right).$$

Remark 6.2. Let $x_i : \Sigma X_i \rightarrow \bigvee_{i=1}^j \Sigma X_i$ denote the i -th summand inclusion. Then the $(j-1)!$ maps

$$\Sigma(X_1 \wedge \cdots \wedge X_j) \rightarrow \bigvee_{i=1}^j \Sigma X_i$$

which describe the inclusion of the homotopy fiber are indexed by the standard monomials of degree j in the Hall basis for the free Lie algebra on the symbols x_1, \dots, x_j . Each such monomial is a basic commutator and the map associated to this basic commutator is given by the r -fold Whitehead product corresponding to it. For example, when $r = j$, the basic commutators of degree 3 in the Hall basis are $[[x_3, x_2], x_1]$ and $[[x_3, x_1], x_2]$.

If X_\bullet is the wedge cube defined by X_1, \dots, X_j , then recall $C(X_\bullet)$ is the homotopy cofiber of the map

$$\bigvee_{i=1}^j \Sigma X_i \rightarrow \operatorname{holim}_{S \neq \mathcal{J}} \left(\bigvee_S \Sigma X_S \right).$$

Then the sequence of 6.1 gives rise to a transgression map

$$(5) \quad \bigvee_{(j-1)!} \Sigma^2(X_1 \wedge \dots \wedge X_j) \rightarrow C(X_\bullet).$$

A straightforward application of the dual Blakers-Massey theorem as applied to the sequence of 6.1 yields

Corollary 6.3. *Assume X_i is r_i -connected for $i \in \mathcal{J}$. The map (5) is $(2 + j + \Sigma + \mu)$ -connected.*

Proof of 3.13. Let A_\bullet be a strongly cocartesian j -cube in which A_\emptyset is a point. The connectivity hypothesis of 3.13 implies that A_i is r_i -connected for $1 \leq i \leq j$.

Then the cube of suspensions SA_\bullet is strongly cocartesian. Apply reduced suspension to this cube to get another strongly cocartesian j -cube ΣSA_\bullet . By Lemma 3.6 we infer that $S^2 A_\bullet$ is weak equivalent to a wedge cube B_\bullet on the spaces $B_i = S^2 A_i$. Then B_i is $(r_i + 2)$ -connected.

Applying 6.3, we get a map

$$(6) \quad \bigvee_{(j-1)!} \Sigma^2(SA_1 \wedge \dots \wedge SA_j) \rightarrow C(S^2 A_\bullet)$$

which is $(4 + 3j + \Sigma + \mu)$ -connected.

Recall that the collection $\{C(S^k A_\bullet)\}_{k \geq 0}$ forms a spectrum $\mathcal{C}(A_\bullet)$ with structure maps $\Sigma^j C(S^k A_\bullet) \rightarrow C(S^{k+1} A_\bullet)$. With respect to this, the map (6) underlies a map of spectra

$$V_j \wedge (SA_1 \wedge \dots \wedge SA_j) \rightarrow \mathcal{C}(S^2 A_\bullet)$$

where V_j is the spectrum given by the wedge of $(j-1)!$ copies of the sphere spectrum S^2 . Desuspend this map $2j$ times to get a $(4 + j + \Sigma + \mu)$ -connected map of spectra

$$(7) \quad W_j \wedge (SA_1 \wedge \dots \wedge SA_j) \xrightarrow{\sim} \mathcal{C}(A_\bullet)$$

where W_j is the wedge of $(j-1)!$ copies of S^{2-2j} . Here we are using the identification of spectra $\Sigma^{2j} \mathcal{C}(A_\bullet) \simeq \mathcal{C}(S^2 A_\bullet)$. This completes the argument when $X = A_\emptyset$ is a point.

When X is arbitrary the argument above is identical except that we need to produce a version of the map (7) in the fibered case. For this to work, we need to write down a fiberwise version of the Whitehead

product map which was discussed in 6.2. We will sketch one way to do this without going into much detail. Without loss in generality we can assume that $X = BG$ for some topological group G . The basic idea is then that $\mathcal{R}(X)$ can be identified (up to Quillen equivalence) with the model category of based G -spaces in which a weak equivalence and fibration are defined by applying the forgetful functor to spaces. In one direction, the equivalence is given by assigning to a based G -space its Borel construction.

It is therefore sufficient to write down Whitehead product maps in based G -spaces, and then transfer over to $\mathcal{R}(X)$ by taking the Borel construction. If Y and Z are based G -spaces then we only need to observe that the usual Whitehead product map $\Sigma Y \wedge Z \rightarrow \Sigma Y \vee \Sigma Z$ of the underlying spaces of Y and Z is obviously equivariant. This does what we want. \square

Remark 6.4. Another proof of 3.13 follows from a fiberwise version of an argument given by Barratt and Whitehead [BW]. We explain the method without giving details. Let B_\bullet be the cube with $B_T = \text{hocolim}(X \leftarrow A_{\mathcal{J}} \rightarrow A_T)$. It is straightforward to check that B_T is weak equivalent to a fiberwise wedge of the objects $\Sigma_X B_i$ for $i \notin T$. With respect to this identification, B_\bullet is a backwards wedge cube.

There is an evident map of cubes $A_\bullet \rightarrow B_\bullet$ which by the Blakers-Massey theorem for $(n+1)$ -cubes is $(1+\mu+\Sigma)$ -cartesian. Consequently, the square

$$\begin{array}{ccc} A_{\mathcal{J}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{holim}_{S \neq \mathcal{J}} A_S & \longrightarrow & \text{holim}_{S \neq \mathcal{J}} B_S \end{array}$$

is $(1+\mu+\Sigma)$ -cartesian. The final step is to analyse $\text{holim}_{S \neq \mathcal{J}} B_S$ as space over X in the stable range using the Hilton-Milnor theorem. One sees that it coincides with $\mathcal{W}_j \wedge_X \left(\bigwedge_{i \in \mathcal{J}} S_X A_i \right)$.

7. MULTIPLE DISJUNCTION

Let $\text{emb}(P, N)$ denote the space of smooth embeddings from P to N . Then $S \mapsto \text{emb}(P, N - Q_S)$ forms a j -cube of spaces, denoted $\text{emb}(P, N - Q_\bullet)$. The natural transformation from embeddings to functions, $\text{emb}(P, N - Q_\bullet) \rightarrow \text{map}(P, N - Q_\bullet)$ is a map of j -cubes, which will be considered as a $(j+1)$ -cube. One of the main results of [GK] is

Theorem 7.1. *Assume $p, q_i \leq n-3$. Then the $(j+1)$ -cube $\text{emb}(P, N - Q_\bullet) \rightarrow \text{map}(P, N - Q_\bullet)$ is $(n-2p-1+\Sigma)$ -cartesian.*

Let $f \in \operatorname{holim}_{S \neq \mathcal{J}} \operatorname{emb}(P, N - Q_S)$. Then f is represented by a family of embeddings $\Delta^{j-1} \rightarrow \operatorname{emb}(P, N)$ such that the image of $d_i \Delta^{j-1}$ is contained in $\operatorname{emb}(P, N - Q_{i+1})$ for $i = 0, 1, \dots, j-1$. By forgetting information, we may also regard f as a map $P \rightarrow \operatorname{holim}_{S \neq \mathcal{J}} N - Q_S$.

Corollary 7.2 (Multiple disjunction). *Assume $p, q_i \leq n-3$ and*

$$p \leq 1 + \min(\mu, n - p - 2) + \Sigma.$$

Suppose $\chi(f) = 0$. Then there is a path in $\operatorname{holim}_{S \neq \mathcal{J}} \operatorname{emb}(P, N - Q_S)$ from f to g , where $g: \Delta^{j-1} \rightarrow \operatorname{emb}(P, N)$ has image inside $\operatorname{emb}(P, N - Q_{\mathcal{J}})$.

8. A CONJECTURE

In [KW1, §13] and [KW2], we showed described a \mathbb{Z}_2 -symmetrized version of the invariant $\chi(f)$ giving the complete obstruction to deforming an immersion $f: P \rightarrow N$ to embedding in the metastable range. The goal of this section is to produce a conjecture generalizing this which is valid beyond the metastable range, under an additional codimension ≥ 3 assumption. The conjecture will be phrased in terms of the Weiss embedding tower of the embedding functor. This is a tower of fibrations of the form

$$\cdots \rightarrow E_2(P, N) \rightarrow E_1(P, N)$$

such that $E_1(P, N)$ is identified with the space of immersions $\operatorname{imm}(P, N)$ and the inverse limit $\lim_j E_j(P, N)$ has the homotopy type of the space of embeddings $\operatorname{emb}(P, N)$ provided $p \leq n-3$.

The spectrum $W_j := \vee_{(j-1)!} S^{2-2j}$ appearing in previous sections the $(j-1)$ -fold desuspension of the j -th coefficient spectrum of the identity functor, in the sense of Goodwillie's homotopy functor calculus. Denote this spectrum by C_j . It has an action of the symmetric group Σ_j and has the nonequivariant homotopy type of the wedge $(j-1)!$ copies of the $(1-j)$ -sphere. Others have noted that the Σ_j -action on C_j extends to a Σ_{j+1} -action. We will make use of this.

Let $f: P \rightarrow N$ be a map. Let $E_j(f)$ be the $(j+1)$ -fold homotopy fiber product of P over N . Then Σ_j acts on $E_j(f)$. Let $D_j(f)$ be the homotopy pullback of the diagram

$$\Delta_j^{\text{fat}}(P) \longrightarrow \prod_j N \xleftarrow{\Delta} N$$

where $\Delta_j^{\text{fat}}(P)$ denotes the fat diagonal of P in $\prod_j P$. Define $E'_j(f)$ to be the space obtained by deleting $D_j(f)$ from $E_j(f)$. Then $E'_j(f)$ inherits a Σ_j -action. Let ξ denote the virtual bundle $(j-1)(\tau_N) - j\tau_P$.

Then ξ pulls back to a virtual bundle on $E'_j(f)$, this bundle admits a Σ_j -action. The smash product $W_{j-1} \wedge E'_j(f)^\xi$ therefore comes equipped with a Σ_j -action (here we are using the action of Σ_j on C_{j-1} alluded to above). Now form the homotopy orbit spectrum

$$C_{j-1} \wedge_{h\Sigma_j} E'_j(f)^\xi.$$

Conjecture 8.1. *Assume $j \geq 2$. Suppose that $f: P \rightarrow N$ is an immersion which comes equipped with a lift $f_{j-1} \in E_{j-1}(P, N)$. Then there is an obstruction*

$$\mu(f_{j-1}) \in \pi_{j-1}(C_{j-1} \wedge_{h\Sigma_j} E'_j(f)^\xi)$$

that vanishes when f_{j-1} lifts to $E_j(P, N)$.

Conversely, assume $(j+1)p + 2j - 1 \leq jn$ and $p \leq n - 3$. Then the vanishing of $\mu(f_{j-1})$ implies that f_{j-1} lifts to an embedding of P in N .

Remark 8.2. When $j = 2$ the conjecture is true and amounts to the bordism obstructions to deforming an immersion to an embedding in metastable range (cf. [HQ, th. 2.3], [KW1, th. A.4]).

The transfer map

$$\mathrm{tr}: \pi_{j-1}(C_{j-1} \wedge_{h\Sigma_j} E'_j(f)^\xi) \rightarrow \pi_{j-1}(C_{j-1} \wedge E'_j(f)^\xi) = \bigoplus_{(j-2)!} \Omega_{2j-2}(E'_j(f); \xi)$$

maps the obstruction group of 8.1 into a direct sum of copies of a normal bordism group. Assuming the conjecture holds, it would be interesting to have a geometric interpretation of $\mathrm{tr}(\mu(f_{j-1}))$. When $j = 2$, one obtains in this way the classical formula for an immersion that relates its self-intersection class to its the normal Euler characteristic and the bordism class of the double point manifold (see e.g., [HQ, p. 338], [KW1, th. A.7]). Note that the transfer is injective after tensoring with the rational numbers. Therefore $\mathrm{tr}(\mu(f_{j-1}))$ detects $\mu(f_{j-1})$ modulo torsion.

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